1. A student sets up a reaction to reflux at 300 deg C, at 9 P.M. when the room temperature was 25 deg C, and goes home. She returns at 5 A.M. and finds that the power had failed during the night and the temperature of the reaction had dropped to 80 deg C. Waiting another 1h without power, the temperature dropped to 50 deg C. Assuming that the room temperature drops 1/2 deg every hour from 9 P.M. to 6 A.M., set up and solve a differential equation to determine the time at which power failed.

Ans:

Newton's law of cooling: $dT/dt = -k(T - T_R)$

The room temperature, $T_R = 25 - 0.5t$, $t$ = 0 at 9 P.M. ($t$ = 8 at 5 A.M, 9 at 6 A.M.)

$\frac{dT}{dt} + kT = 25k - 0.5kt$ (inhomogeneous I order equation)

The homogeneous equation $\frac{dT}{dt} + kT = 0$ has the general solution, $T = A\exp(-kt)$

Consider a particular solution of the full equation in the form $Bt + C$, and obtain by substitution, $B = -0.5$ and $C = 25 + 0.5(1/k)$, so that its general solution is

$T = A\exp(-kt) - 0.5t + 25 + 0.5(1/k)$

Find $A$ and $k$ by requiring that $T$ = 80 at $t$ = 8; $T$ = 50 at $t$ = 9.

$A\exp(-8k) + 0.5(1/k) + 21 = 80; A\exp(-9k) + 0.5(1/k) + 20.5 = 50$

$A\exp(-8k) + 0.5(1/k) = 59$

$A\exp(-8k) + 0.5(1/k) = 29.5$

$A\exp(-8k) = 59 - 0.5/k$

$A\exp(-9k) = 29.5 - 0.5/k$

$\log A - 8k = \log(59 - 0.5/k)$

$\log A - 9k = \log(29.5 - 0.5/k)$

Solve for $A$ and $k$

Find $t$ when $T$ = 300 K for the function $T = A\exp(-kt) - 0.5t + 0.5(1/k) + 25)$

2. For the process, $A \xrightarrow{k_a} B \xrightarrow{k_b} C$

set up and solve an equation to give the amount of $B$ at any time $t$.

Ans:

$\frac{dA}{dt} = -k_a A; \quad \frac{dB}{dt} = -k_b B + k_a A$

$\frac{dB}{dt} + k_b B = k_a A_0 \exp(-k_a t)$

$dB + \{k_b B - k_a A_0 \exp(-k_a t)\}dt = 0$

Not an exact differential equation. Verify that it can be made exact by multiplying with $\exp(k_b t)$

The eqn becomes

$$e^{k_b t} \frac{dB}{dt} + k_b B e^{k_b t} = k_a A_0 e^{(k_b - k_a)t}$$

$$\frac{d(Be^{k_b t})}{dt} = k_a A_0 e^{(k_b - k_a)t}$$

$$\therefore B = e^{-k_b t} k_a A_0 \int e^{(k_b - k_a)t} = e^{-k_b t} \{ \frac{k_a A_0}{k_b - k_a} e^{(k_b - k_a)t} + c \}$$

Assign the constant $C$, by noting $B = 0$ at $t = 0$. Simplify.

3. Another way of writing Hermite polynomial (different from that shown in the class) is as follows:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

Use the above equation to write down the first six Hermite polynomials and verify the following general form:

$$H_n(x) = (2x)^n - \frac{n(n-1)(2x)^{n-2}}{1!} + \frac{n(n-1)(n-2)(n-3)(2x)^{n-4}}{2!} + \ldots$$

4. Show by substitution that the function, $e^{-x^2/2} H_n(x)$ where, $H_n(x)$ denotes Hermite polynomials, satisfies the differential equation $y'' + (1 - x^2 + 2n)\, y = 0$. Plot this function for $n = 0$, $n = 1$ and $n = 2$.