CY404: Assignment - 2

<u>Go through the following worked exercises and spot errors, if any</u>:

1. Evaluate indefinite integrals:

(a) $x \sin x$

Ans: $\int x \sin x dx = -x \cos x - \int -\cos x dx = -x \cos x + \sin x$

(b) $x^3 e^{-x^2}$

Ans: Let $x^2 = t$. Then $2xdx = dt$

$$\int x^3 e^{-x^2} dx = \frac{1}{2}\int te^t dt = \frac{1}{2}[te^t - \int e^t dt] = \frac{1}{2}(te^t - e^t) = \frac{1}{2} e^{x^2}(x^2 - 1)$$

(c) $\frac{1}{4 - 4x^2 + 6x}$

Ans: $4 - 4x^2 + 6x = -\left(2x - \frac{3}{2}\right)^2 + \frac{25}{4}$

Let $2x - \frac{3}{2} = t.$ Then, $dx = dt/2$

$$\int \frac{dx}{4 - 4x^2 + 6x} = \frac{1}{2}\int \frac{dt}{\left(\frac{5}{2}\right)^2 - t^2} = \frac{1}{2}\frac{2}{5}\tanh^{-1}\frac{2}{5}t = \frac{1}{5}\tanh^{-1}\frac{2t}{5} = \frac{1}{5}\tanh^{-1}\frac{4x - 3}{5}$$

(We used one of the standard integrals, $\int \frac{dx}{a^2 - x^2} = \frac{\tanh^{-1}\frac{x}{a}}{a}$)

(d) $\frac{1}{4\cos x + 3\sin x}$

Ans: Let $\tan\frac{x}{2} = t$

$$\int \frac{dx}{4\cos x + 3\sin x} = \int \frac{2dt}{(1+t^2)\left(\frac{4(1-t^2)}{1+t^2} + \frac{6t}{1+t^2}\right)} = \int \frac{2dt}{4 - 4t^2 + 6t} = \frac{2}{5}\tanh^{-1}\frac{4t-3}{5}$$

(the last result from problem (c))

The require integral $= \frac{2}{5}\tanh^{-1}\frac{4\tan\frac{x}{2} - 3}{5}$

2. Evaluate definite integrals:

(a) $\int_0^\infty e^{-x}\sin x dx$

Use *udv* rule with *u* = sin *x* and *dv* = e^{-x}

$$\int e^{-x}\sin x dx = -\sin x e^{-x} - \int -e^{-x}\cos x dx = -\sin x e^{-x} + -\cos x e^{-x} - \int -e^{-x}(-\sin x)dx$$

$$= -\sin x e^{-x} - \cos x e^{-x} - \int e^{-x}\sin x dx$$

$$\therefore 2\int e^{-x}\sin x dx = -\sin x e^{-x} - \cos x e^{-x}$$

$$\therefore \int e^{-x}\sin x dx = -\frac{1}{2}e^{-x}(\sin x + \cos x)$$

Applying limits, the required definite integral is ½. (becuase, at upper limit the integral is zero and at lower limit it is cos 0 = 1.

(b) $\int_0^{\pi/2} x\cos^2 x dx$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\int x\cos^2 x dx = \frac{1}{2}\int x dx + \frac{1}{2}\int x\cos 2x dx$$

$$\int x dx = \frac{x^2}{2}$$

$$\int x\cos 2x dx = x\frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} dx = \frac{1}{2}x\sin 2x - \frac{1}{2}(-\frac{\cos 2x}{2}) = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x$$

$$\therefore \int x\cos^2 x dx = \frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8}$$

Applying limits, the required definite integral is

$$\frac{\pi^2}{16} - 0 + \frac{\pi}{2}\frac{\sin\pi}{4} - \frac{\pi}{2}\frac{\sin 0}{4} + \frac{\cos\pi}{8} - \frac{\cos 0}{8} = \frac{\pi^2}{16} - 0 + 0 - 0 - \frac{1}{8} - \frac{1}{8} = \frac{\pi^2}{16} - \frac{1}{4}$$

(c) $\int_0^x \sin^2 y dy$

Ans: $\int \sin^2 y dy = \frac{1}{2}\int (1 - \cos 2y) dy = \frac{1}{2}(y - \frac{\sin 2y}{2}) = \frac{y}{2} - \frac{\sin 2y}{4}$

Applying limits, the required definite integral is the function, $\frac{x}{2} - \frac{\sin 2x}{4}$

3. Evaluate $\int_0^t e^x \cos x dx$ by using complex number to replace the trignometric function

Ans: $\int_0^t e^x \cos x dx + i\int_0^t e^x \sin x dx =$

$$\int_0^t e^x (\cos x + i\sin x) dx = \int_0^t e^x e^{ix} dx = \int_0^t e^{x(1+i)} dx$$

$$= \frac{e^{t(1+i)} - 1}{1+i} = \frac{e^t(\cos t + i\sin t) - 1}{1+i}\frac{1-i}{1-i} = \frac{\{e^t(\cos t + i\sin t) - 1\}(1-i)}{2}$$

The required definite integral is the real part of the above complex quantity, which is

$$\frac{e^t(\cos t + \sin t) - 1}{2}$$

Note: Using complex number to replace sine or cosine sometimes simplifies the algebra.

4. What angle does the body diagonal of a cube make with one of its edges?

Ans: Unit vector along body diagonal = $\frac{1}{\sqrt{3}}(\vec{i}+\vec{j}+\vec{k})$

Unit vector along an edge (let us call it the *x*-edge) = $\vec{i}$

$$\frac{1}{\sqrt{3}}(\vec{i}+\vec{j}+\vec{k})\cdot\vec{i}=\frac{1}{\sqrt{3}}$$

$$\text{Angle}=\cos^{-1}(\frac{1}{\sqrt{3}})=54.73\,\text{degrees}$$

5. Find $\nabla^2\left(\frac{x\cos xy}{z}\right)$

Ans:

$$\frac{\partial f}{\partial x}=\frac{\cos xy}{z}-\frac{xy\sin xy}{z};\quad \frac{\partial^2 f}{\partial x^2}=\frac{-y\sin xy}{z}-\frac{xy^2\cos xy}{z}-\frac{y\sin xy}{z}$$

$$\frac{\partial f}{\partial y}=\frac{-x^2\sin xy}{z};\quad \frac{\partial^2 f}{\partial y^2}=\frac{-x^3\cos xy}{z}$$

$$\frac{\partial f}{\partial z}=\frac{-x\cos xy}{z^2};\quad \frac{\partial^2 f}{\partial z^2}=\frac{2x\cos xy}{z^3}$$

$$\therefore \nabla^2 f=-\frac{y\sin xy}{z}-\frac{xy^2\cos xy}{z}-\frac{y\sin xy}{z}-\frac{x^3\cos xy}{z}+\frac{2x\cos xy}{z^3}$$

6. Evaluate $\oint_C(2xydy-x^2dx)$ where the cyclic path C is the triangle

A(0,0) -> B(1,0) -> C(1,1) -> A(0,0)

A -> B $y = 0$ $dy = 0$ $\int_0^1 -x^2dx=-1/3$

B -> C $x = 1$ $dx = 0$ $\int_0^1 2y\,dy = 1$

C -> A $x = y$ $\int_1^0 x^2 = -1/3$

The net result is therefore, –1/3 + 1 –1/3 = 1/3

Try the following problem:

A vector function is given as $\vec{f} = x^2\ \vec{i} + xy\ \vec{j} + z\ \vec{k}$. Find the integral $\int_C \vec{f} \cdot d\,\vec{r}$ along a straight line joining the origin to the point (1, 1, 0).

If the vector function in the above calculation represents a force which is derived from a potential, $V(x, y, z)$ such that $\vec{f} = \vec{\nabla} V(x, y, z)$, show that the above integral does not depend on the path. What will be the value of the integral if the displacement is from (x_1, y_1, z_1) to (x_2, y_2, z_2)?