CY404: Assignment - 1

Go through the following worked exercises and spot errors, if any:

1. If $f(x) = 2x^2 - 1$, what is $f(\sin t)$?

Ans: $2\sin^2 t - 1 = -\cos 2t$

2. $e^{x^2} \cos 3x$ is an even function. True or False?

Ans: True (because both e^{x^2} and $\cos 3x$ are even).

What about $e^x \cos 3x$ and $e^{x^2} \sin 3x$?

3. Differentiate

(a) $\log\cos(\frac{1}{x})$

Ans: $\frac{1}{\cos\frac{1}{x}}(-\sin\frac{1}{x})(-\frac{1}{x^2}) = \frac{1}{x^2}\tan\frac{1}{x}$

(b) $\sin(\cos x)$

Ans: $\{\cos(\cos x)\}(-\sin x) = -\sin x\cos(\cos x)$

(c) $x^{\cos x}$

Ans: $x^{\cos x - 1}(-\sin x) = -\sin x \; x^{\cos x - 1}$

(d) $\sin^{-1}(\frac{x}{x+1})$

Ans: let $\sin^{-1}(\frac{x}{x+1}) = t$ so that, $\sin t = \frac{x}{x+1}$

$$\cos t \frac{dt}{dx} = \frac{1}{x+1} - \frac{x}{(x+1)^2}$$

Therefore, the required derivative is

$$(\frac{1}{\cos t})\{\frac{1}{x+1}-\frac{x}{(x+1)^2}\}=\frac{\frac{1}{x+1}-\frac{x}{(x+1)^2}}{\sqrt{1-(\frac{x}{x+1})^2}}=\frac{1}{(x+1)\sqrt{2x+1}}$$

4. If $z = x + iy$ what is log(z)?

Ans: $z = re^{i\theta}; \quad \log(z) = \log(r) + i\theta = \log(\sqrt{x^2 + y^2}) + i(\tan^{-1}\frac{y}{x})$

5. Find the real and imaginary part of sin(ix)

Ans: $e^{ix} = \cos x + i \sin x; \quad e^{-ix} = \cos x - i\sin x; \quad e^{-ix} - e^{ix} = -2i \sin x$

$$\sin x = -\frac{1}{i}\frac{(e^{-ix} - e^{ix})}{2} = i\frac{(e^{-ix} - e^{ix})}{2} \quad \therefore \sin ix = i\frac{(e^{x} - e^{-x})}{2} = i\sinh x$$

Real part = 0; imaginary part = $i$sinh x

6. Show that $(\cos x + i \sin x)^2 = \cos 2x + i \sin 2x$

Ans: Let $Z = e^{ix}$. Then, $(\cos x + i\sin x)^2 = Z^2 = e^{2ix} = \cos 2x + i \sin 2x$

Generalise this to de Moivre’s theorem: $(\cos x + i\sin x)^n = \cos nx + i \sin nx$ for n = positive or negative integer or a fraction p/q where p and q are integers.

<u>Try the following problems</u>:

1. Consider the two functions:

(i) Gaussian: $\exp(-a^2x^2)$

(ii) Lorentzian: $1/(a^2 + x^2)$.

For each function derive an expression for the half width (*ie*., the width at half height).

Plot the two functions and their first derivatives for a =1.

2. If $u = (x^2-1)^n$, where n is a positive integer, show that,

$$u^k = (n-k+1)(2x)\frac{u^{k-1}}{x^2-1} + (k-1)(2n-k+2)\frac{u^{k-2}}{x^2-1}$$

where, u^k denotes the k^{th} derivative of u.

3. The complex number $re^{i\theta}$ may be symbolically written as (r,θ). Show that,

$$[(r,\theta)+(s,\varphi)](t,\psi) = (r,\theta)(t,\psi)+(s,\varphi)(t,\psi).$$