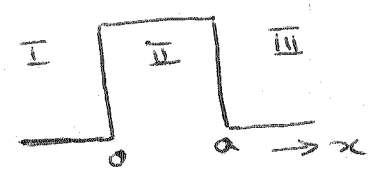
15. Obtain an expression for the tunneling frequency across a 1-dimensional potential barrier.



Region I $\quad \dfrac{d^2\psi}{dx^2} + k^2\psi = 0$

$$k^2 = 2mE/\hbar^2$$

$$\psi^{I} = A_1 e^{ikx} + B_1 e^{-ikx}$$

Region II $\quad \dfrac{d^2\psi}{dx^2} + k_2^2\psi = 0$

$$k_2^2 = \frac{2m(E-V)}{\hbar^2} \Rightarrow k_2 = i k_3 \quad \text{when } E < V$$

$$\psi^{II} = A_2 e^{ik_2x} + B_2 e^{-ik_2x}$$

Region III $\quad \psi^{III} = A_3 e^{ikx} + B_3 e^{-ikx}$

Continuity at the walls $(x=0, x=a)$ leads to 4 equations, for the wave function.

for the 6 parameters in the region III

Put $B_3 = 0$ (arbitrary condition ; particle in only moves to the right)

$$A_1 + B_1 = A_2 + B_2$$

$$A_1 - B_1 = (A_2 - B_2)\, q \qquad q = k_2/k$$

$$A_2 e^{ik_2a} + B_2 e^{-ik_2a} = A_3 e^{ika}$$

$$A_2 e^{ik_2a} - B_2 e^{-ik_2a} = \frac{A_3}{q} e^{ika}$$

Define $T = |A_3|^2/|A_1|^2$ as the tunneling frequency.

The ~~abo~~ four equations lead to (Show how!)

$$\frac{A_1}{A_3} = \frac{e^{ika}}{4q}\left\{(q^2+2q+1)\,e^{-ik_2 a} + (-q^2+2q-1)\,e^{ik_2 a}\right\}$$

$$= \frac{e^{ika}}{4q}\left\{u\,e^{-ik_2 a} + v\,e^{ik_2 a}\right\}$$

$$= \frac{e^{ika}}{4q}\left(u\,e^{\kappa a} + v\,e^{-\kappa a}\right)$$

$$|A_1|^2/|A_3|^2 = \frac{1}{16qq^*}\left(uu^*\,e^{2\kappa a} + vv^*\,e^{-2\kappa a} + uv^* + vu^*\right)$$

$$\sim e^{2\kappa a}$$ (omitting the smaller terms in first approximation)

$$\therefore \quad T \sim e^{-2\kappa a} = e^{-\frac{2}{\hbar}\sqrt{2m(V-E)}\,a}$$

$T \to 0$ when $a \to \infty$ or $m \to \infty$ or $V \to \infty$

Show that the exact result is

$$T = \frac{1}{\left(\frac{1}{4}+\delta\right)e^{2\kappa a} + \left(\frac{1}{4}+\delta\right)e^{-2\kappa a} + \left(\frac{1}{2}-\frac{\delta}{2}\right)}$$

where $\delta = (\kappa^2 - k^2)^2/16\kappa^2 k^2$