Tutorial Notes #3. Degeneracy..

* Let us illustrate with 2-fold degeneracy (may be generalized)

$\hat{Q}\,\phi_1 = q\,\phi_1 \qquad \hat{Q}\,\phi_2 = q\,\phi_2$

In this case ϕ_1 and ϕ_2 need not be orthogonal (recall "eigenfunctions of a Hermitian operator belonging to DIFFERENT eigenvalues are orthogonal")

There are ways of making ϕ_1 orthogonal to ϕ_2. If $\langle \phi_1 | \phi_2 \rangle = S$, and assuming that ϕ_1 and ϕ_2 are normalised, it is easy to see that the combination $\phi' = \phi_1 - S\phi_2$ is orthogonal to ϕ_2. Note that the eigenvalues remain the same for ϕ_1' and ϕ_2.

$\hat{Q}\,\phi_1' = \hat{Q}\,\phi_1 - \hat{Q}(S\phi_2) = q\,\phi_1 - S\,q\,\phi_2 = q(\phi_1 - S\phi_2).$

* In general, we can chose any linear combination form a degenerate set.

$$\left.\begin{aligned} \phi_1' &= a_1\,\phi_1 + a_2\,\phi_2 \\ \phi_2' &= b_1\,\phi_1 + b_2\,\phi_2 \end{aligned}\right\} \text{ eigen value } = q$$

The coefts may be chosen such that ϕ_1' and ϕ_2' form an ortho-normal set.

★ Linear dependence: In the above example, if we make $\phi_3' = c_1 \phi_1 + c_2 \phi_2$, then one can show that $\phi_3' = a\,\phi_1' + b\,\phi_2'$ – by substitution you may express a and b in terms of a_1, a_2, b_1, b_2.

We say that ϕ_3' is <u>not</u> linearly independent of ϕ_1' and ϕ_2'.

In general for an n-fold degenerate state one can construct n linearly independent combinations, ~~chosen~~ with coefts chosen such that they from an orthonormal set.

★ Effect of commuting operators ~~on~~ on a degenerate set:

Let $\hat{P}$ and $\hat{Q}$ be two commuting operators and let ϕ be an eigen function of $\hat{Q}$

$\hat{Q}\,\phi = q\,\phi$

Since $\hat{P}\hat{Q} = \hat{Q}\hat{P}$, $\hat{Q}\,\hat{P}\,\phi = \hat{P}\,\hat{Q}\,\phi = q\,\hat{P}\,\phi$

What does $\hat{Q}(\hat{P}\,\phi) = q(\hat{P}\,\phi)$ imply?

If ϕ is an non-degenerate eigen function of $\hat{Q}$, it would imply $\hat{P}\,\phi = p\,\phi$, where p is a constant which we identify as the eigenvalue of $\hat{P}$ for ϕ.

Since ϕ is one of the degenerate eigenfunctions of $\hat{Q}$, there is a second possibility: $\hat{P}\phi$ may be any member or a linear combination of several members of the degenerate set.

For our two fold degeneracy example.

$\hat{Q}\phi_1 = q\phi_1 \qquad \hat{Q}\phi_2 = q\phi_2$

$\hat{P}\hat{Q} = \hat{Q}\hat{P}$ implies $\hat{P}\phi_1 = a_1\phi_1 + a_2\phi_2$.

$\therefore \phi_1$ ~~need not be~~ will be an eigenfunction of $\hat{P}$ only if $a_2 = 0$. Note, $a_1\phi_1 + a_2\phi_2$ is still an eigenfunction of $\hat{Q}$.

★ Example: $\hat{Q} = \dfrac{d^2}{dx^2}$ and $\hat{P} = \dfrac{d}{dx}$

$\cos x$ and $\sin x$ are two degenerate eigen function of $\hat{Q}$ (eigen value $= -1$). They are <u>not</u> eigen function of $\hat{P}$. But their linear combinations $\cos x + i\sin x$ and $\cos x - i\sin x$ are eigen functions (with eigen value $+i$ and $-i$.) Note that there is no degeneracy for $\hat{P}$.

What is the common complete set of eigen function of $\hat{P}$ and $\hat{Q}$?

★ Consider another example $\hat{Q} = \hat{1}$ $\hat{P} = \dfrac{d}{dx}$. What happens in this case? $(\hat{1}\phi = 1\phi)$