Tutorial Notes #2 Angular momentum

☆ $\hat{L}^2 \,|l, m\rangle = l(l+1)\,|l, m\rangle \qquad \hat{L}_z\,|l, m\rangle = m\,|l, m\rangle$

$$\left.\begin{aligned} \hat{L}_x\,|l, m\rangle &= \tfrac{1}{2}\left\{\hat{L}_+\,|l, m\rangle + \hat{L}_-\,|l, m\rangle\right\} \\ \hat{L}_y\,|l, m\rangle &= \tfrac{1}{2i}\left\{\hat{L}_+\,|l, m\rangle - \hat{L}_-\,|l, m\rangle\right\} \end{aligned}\right\} \text{ because, } \hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$$

Example $\hat{L}_x\,|3, 3\rangle = \tfrac{1}{2}\left\{\hat{L}_+\,|3, 3\rangle + \hat{L}_-\,|3, 3\rangle\right\}$

$$\cancel{= \tfrac{1}{2}\ 3(3} = \tfrac{1}{2}\left[0 + \sqrt{3(3+1) - 3(3-1)}\ |3, 2\rangle\right]$$

$$= \sqrt{\tfrac{3}{2}}\ |3, 2\rangle$$

$$\hat{L}_x\,|3, 2\rangle = \sqrt{\tfrac{3}{2}}\ |3, 3\rangle + \sqrt{\tfrac{5}{2}}\ |3, 1\rangle \qquad \text{Check!}$$

Note that $\langle l, m|\hat{L}_x|l, m\rangle = 0$!

Practise: find $\hat{L}_y\,|\tfrac{5}{2}, -\tfrac{1}{2}\rangle$

☆ Examples of angular momentum matrices:

$\hat{L}_x$	$\|1,-1\rangle$	$\|1,0\rangle$	$\|1,1\rangle$
$\langle 1,-1\|$	0	$1/\sqrt{2}$	0
$\langle 1,0\|$	$1/\sqrt{2}$	0	$1/\sqrt{2}$
$\langle 1,1\|$	0	$1/\sqrt{2}$	0

Real Symmetric

$\hat{L}_y$	$\|1,-1\rangle$	$\|1,0\rangle$	$\|1,1\rangle$
$\langle 1,-1\|$	0	$i/\sqrt{2}$	0
$\langle 1,0\|$	$-i/\sqrt{2}$	0	$i/\sqrt{2}$
$\langle 1,1\|$	0	$-i/\sqrt{2}$	0

Hermitian

$\hat{L}_z$	$\|1,-1\rangle$	$\|1,0\rangle$	$\|1,1\rangle$
$\langle 1,-1\|$	-1	0	0
$\langle 1,0\|$	0	0	0
$\langle 1,1\|$	0	0	1

Diagonal

Note that $\hat{L}_x, \hat{L}_y, \hat{L}_z$ are Hermitian operators.

What about $\hat{L}_+$ and $\hat{L}_-$ →

Check by constructing their matrices!

$$\hat{L}_x^2 = \frac{1}{4}(\hat{L}_+ + \hat{L}_-)^2 = \frac{1}{4}(\hat{L}_+^2 + \hat{L}_-^2 + \hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+)$$

$$\langle l,m|\hat{L}_x^2|l,m\rangle = \frac{1}{4}(\langle l,m|\hat{L}_+\hat{L}_-|l,m\rangle + \langle l,m|\hat{L}_-\hat{L}_+|l,m\rangle$$

$$\neq 0 \quad (\text{if } l \neq 0)$$

Practise: find $\xleftarrow{3}$ $\langle 3,2|\hat{L}_x^2|3,2\rangle$

★ In summary, $|l,m\rangle$ are eigenfunctions of $\hat{L}^2$, $\hat{L}_z$ and $\hat{L}_x^2 + \hat{L}_y^2$ $(= \hat{L}^2 - \hat{L}_z^2)$. In any pure $|l,m\rangle$ state, the average values of L_x and L_y, $\langle L_x\rangle$, $\langle L_y\rangle$ are zero, but $\langle \hat{L}_x^2\rangle$ and $\langle \hat{L}_y^2\rangle$ are not zero (unless $l=0$).

★ Consider the mixed state $|\phi\rangle = c_1\,|1/2,-1/2\rangle + c_2\,|1/2\ 1/2\rangle$, where c_1 and c_2 are arbitrary mixing coefficients.

Then, $\langle \phi|\hat{L}_x|\phi\rangle = c_1 c_2$. check this!

★ How does the uncertainty principle work for the state $|0,0\rangle$? When $l=0$, L_x, L_y and L_z are precisely zero!

$$\delta_{L_x}^2\,\delta_{L_y}^2 \geqslant 1/4\{\langle 0,0|\hat{L}_z|0,0\rangle\}^2 = 0$$

The condition - product of uncertainties greater than or equal to zero is not violated!

Note that in $|l,m\rangle$ state $\delta_{L_x}\delta_{L_z} = \delta_{L_y L_z} = 0$ because in such a state L_z is precise, so L_x and L_y should be infinitely uncertain!