Differential Equations:

These are equations ~~that~~ which involve derivatives of a function. The function (or functions) that satisfy the equation constitute the solution. Many physical problems can be ~~to~~ stated in the form of differential equations. We will consider two examples which have some relevance to this course: the vibrating spring and the vibrating string.

The spring problem   For small displacements from the equilibrium position, the restoring force acting on a vibrating spring is given by the Hooke's law (Such oscillations are said to be harmonic vibrations): $m\dfrac{d^2x}{dt^2} = -kx$, where, x is the displacement, m is the mass and k a constant characteristic of the spring. (Recall, $F = ma$, the acceleration, 'a' being the second derivative of the displacement with respect to time. **(What is velocity ?)**

The differential equation is then,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \quad \text{where } \omega^2 = k/m.$$

(Verify the solutions by direct substitution in the equation)

$x = \sin\omega t$ is a solution of this equation. However, this is not a unique solution, since $x = \cos\omega t$, as well as any linear combination $x = a\cos\omega t + b\sin\omega t$, where a and b are arbitrary constants are also valid solutions. (In general, for a n^{th}-order equation of this type — 'linear, homogeneous equation with constant coefficients' — the most general solution will have n independent constants.) Particular solutions, suitable for a physical problem can be obtained from the general solution by imposing the conditions peculiar to the problem — these are known as boundary conditions.

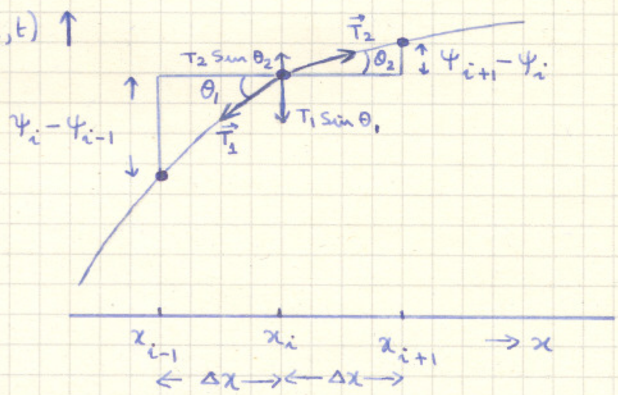
Show that if we impose the two conditions, (i) $x = 0$ when $t = 0$, and (ii) the initial velocity is 2 units, then $x(t) = (2/\omega)\sin(\omega t)$.

The string problem

A string fixed at both ends is set in vibration. The amplitude of vertical displacement is a function of both the horizontal position, x and the time, t. We need an equation connecting the partial derivatives of the displacement, $\psi(x,t)$ with the characteristics of the string.

Consider n point masses (m), elastically connected and spaced at Δx. In the limit $n \to \infty$, $\Delta x \to 0$, $m \to 0$, the ratio $m/\Delta x$ will tend to a constant, μ (the mass per unit length of the string).

The vertical component of force on the ith point mass $= T_2 \sin\theta_2 - T_1 \sin\theta_1$



The magnitudes of $\vec{T_1}$ and $\vec{T_2}$ are equal to the tension in the string

ie $|\vec{T_1}| = |\vec{T_2}| = T$

For small amplitude vibrations, (ie., $\Delta\psi << \Delta x$, $\sin\theta_1 \sim \tan\theta_1$, $\sin\theta_2 \sim \tan\theta_2$

$$\therefore m \frac{\partial^2 \psi}{\partial t^2} = T(\sin\theta_2 - \sin\theta_1)$$

$$= T\left[\frac{(\psi_{i+1}-\psi_i)}{\Delta x} - \frac{(\psi_i - \psi_{i-1})}{\Delta x}\right]$$

$$= T(\psi_{i+1} + \psi_{i-1} - 2\psi_i)/\Delta x$$

To understand the quantity on the right hand side, look at the difference table $\to$

x	ψ	(1) $\Delta\psi/\Delta x$	(2) $\Delta(\Delta\psi/\Delta x)/\Delta x$
x_{i-1}	ψ_{i-1}		
		$\frac{(\psi_i - \psi_{i-1})}{\Delta x}$	
x_i	ψ_i		$\frac{(\psi_{i+1}-\psi_i)-(\psi_i-\psi_{i-1})}{(\Delta x)^2}$
		$\frac{(\psi_{i+1}-\psi_i)}{\Delta x}$	
x_{i+1}	ψ_{i+1}		

$$\therefore m\, \partial^2\psi/\partial t^2 = T \frac{\Delta(\Delta\psi/\Delta x)}{(\Delta x)^2} \cdot \Delta x \to \frac{m}{\Delta x} \frac{\partial^2\psi}{\partial t^2} = T \frac{\Delta(\Delta\psi/\Delta x)}{\Delta x}$$

In the limit $\Delta x \to 0$ the above equation becomes,

$$\mu(\partial^2\psi/\partial t^2) = T\, \partial^2\psi/\partial x^2 \qquad \frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \text{ where,}$$

$c^2 = T/\mu$

This is the partial differential equation describing the vibration of the string. A valid solution for $\psi(x,t)$ should satisfy this equation as well as the boundary condition, $\psi(0) = \psi(L) = 0$ (at all times the ends remain fixed).

A solution may be found by writing,

$$\psi(x,t) = \phi(x)\,\phi(t) \quad \text{(a space part x a time part)}$$

It can be seen that the following are acceptable solutions:

$$\phi(t) = \sin(\omega t) \qquad \phi(x) = \sin\left(\frac{\omega}{c}x\right)$$

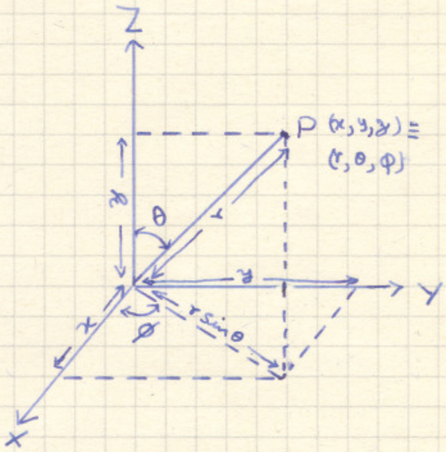
Since $\phi(0) = \phi(L) = 0$, $\frac{\omega}{c} = \frac{n\pi}{L}$, where n is an integer.

Therefore, an acceptable solution will be,

$$\psi(x,t) = \sin\left(\frac{n\pi x}{L}\right)\sin(\omega t)$$

Show that any linear combination $\Psi(x,t) = \sum_{j=1}^{n} a_j \psi_j(x,t)$ is also a valid solution. – This is known as the "principle of superposition"

Cartesian to Polar



$z = r\cos\theta$

$y = r\sin\theta\sin\phi$

$x = r\sin\theta\cos\phi$

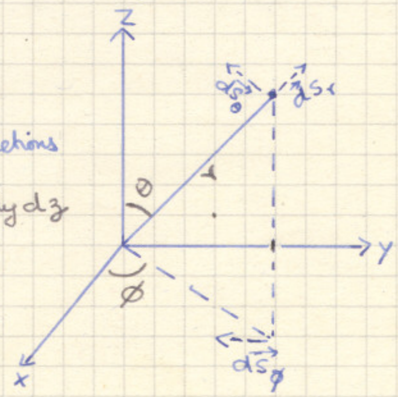
Infinitesimal displacements

$\vec{ds_r} = dr\ \vec{e_r}$

$\vec{ds_\theta} = r\,d\theta\ \vec{e_\theta}$

$\vec{ds_\varphi} = r\sin\theta\,d\phi\ \vec{e_\varphi}$

$\vec{e_r}, \vec{e_\theta}, \vec{e_\varphi}$ are unit vectors along the coordinate directions



$\therefore dV = |ds_r||ds_\theta||ds_\varphi| = r^2\sin\theta\,d\theta\,d\phi\,dr \equiv dx\,dy\,dz$

$\vec{\nabla} = \vec{e_r}\frac{\partial}{\partial s_r} + \vec{e_\theta}\frac{\partial}{\partial s_\theta} + \vec{e_\varphi}\frac{\partial}{\partial s_\varphi}$

$$\therefore\ \vec{\nabla} = \vec{e_r}\frac{\partial}{\partial r} + \frac{\vec{e_\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\vec{e_\varphi}}{r\sin\theta}\frac{\partial}{\partial\phi}$$

It can be further shown that,

$$\nabla^2 f(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

——— x ———