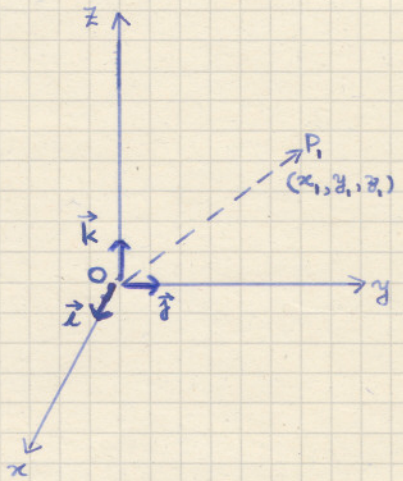
Vectors



(x_1, y_1, z_1) represents the cartesian coordinates of the point P_1. $\vec{OP_1}$ or simply $\vec{P_1}$ is the position vector of the point P_1. If $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors along the cartesian axes x, y, z respectively, the vector $\vec{P_1}$ can be written in terms of its components as follows:

$$\vec{P_1} = x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k}$$

What will be y_1 and z_1 for a point on the x-axis?
What will be x_1 for a point on the yz-plane?
What is the length of the vector $\vec{P_1}$ (ie., the length of the line OP_1)?

If $\vec{P_1}$ makes an angle α_1 with the x-axis, β_1 with the y-axis, and γ_1 with the z-axis, what is the relation between x_1, y_1, z_1 and α_1, β_1, γ_1?

(hint: consider first the simpler 2-dimensional case, ie., a vector confined to the xy plane)

Two products are defined for vectors. If $\vec{a}$ and $\vec{b}$ are two vectors making an angle of θ with each other, then,

the scalar product $= \vec{a} \cdot \vec{b} = a\, b \sin\theta$

the vector product $= \vec{a} \times \vec{b} = ab \sin\theta \ \vec{n}$.

Here a and b are the lengths of $\vec{a}$ and $\vec{b}$ and $\vec{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

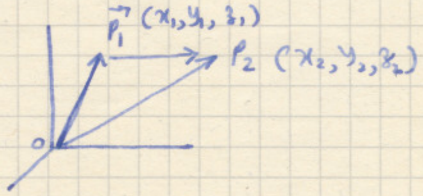
The products between the unit vectors $(\vec{i}, \vec{j}, \vec{k})$ along the cartesian axes are given in the following tables

$\cdot$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	1	0	0
$\vec{j}$	0	1	0
$\vec{k}$	0	0	1

$\times$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	0	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	0	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	0

Calculate the angle between the two vectors, (2,3,0) and (0,1,1).

$$\vec{OP_1} + \vec{P_1P_2} = \vec{OP_2}$$



What is the length of $\vec{P_1P_2}$?

Write $\vec{a} \times \vec{b}$ in terms of its cartesian components.

[Expand $(x_1\vec{i} + y_1\vec{j} + z_1\vec{k}) \times (x_2\vec{i} + y_2\vec{j} + z_2\vec{k})$ using the table of products of unit vectors $\vec{i}, \vec{j}, \vec{k}$.]

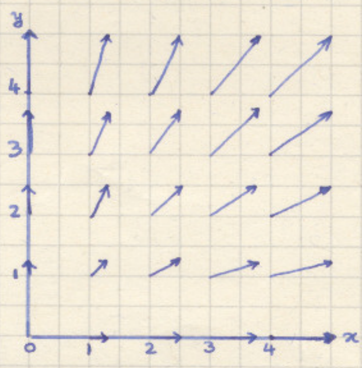
<u>Vector functions</u>: $f_1(x,y,z)\,\vec{i} + f_2(x,y,z)\,\vec{j} + f_3(x,y,z)\,\vec{k}$ is a vector function where $f_1 - f_3$ are ~~coordi~~ functions of coordinate variables.

Examples: $x\vec{i} + y\vec{j}$ $\quad$ ~~x~~ $x\vec{i} + yz\vec{j} + y\vec{k}$

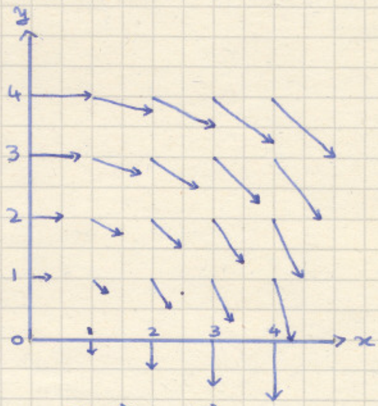
$y\vec{i} - x\vec{j}$ $\quad$ $x^2y\,\vec{i} - y^2z\,\vec{j} + z^2\vec{k}$ etc...

A graphical representation of two of the above functions follow:

{At coordinate grid points vectors are drawn having directions parallel to the vector function at those points. The lengths are scaled down (by 50%) for clarity}



$x\vec{i} + y\vec{j}$



$y\vec{i} - x\vec{j}$

<u>A vector operator called 'del'</u>: $\vec{\nabla} = \vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}$

If f is a scalar function, then $\vec{\nabla} f$ is a vector function called <u>gradient</u>. If $\vec{f}$ is a vector function, then $\vec{\nabla}\cdot\vec{f}$ is a scalar function called <u>divergence</u> and $\vec{\nabla}\times\vec{f}$ is a vector function called <u>curl</u>.

Let us calculate the gradient of x^2+y^2:

$$= \vec{i}\frac{\partial}{\partial x}(x^2+y^2) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2) + \vec{k}\frac{\partial}{\partial z}(x^2+y^2) = 2x\vec{i} + 2y\vec{j}$$

Calculate the gradient of (i) $x + y + z$, (ii) $x^2 - y^2$, (iii) $x^2 + y^2 + z^2$

Calculate the divergence and curl of the two functions plotted above. Are the results 'in agreement with the pictures' ?

What is $\vec{\nabla}\cdot\vec{\nabla}$?

Complex Numbers

These numbers have a real part and an "imaginary part." ~~For~~ In the complex number $a + ib$, a is the real part and ib is the imaginary part, where i stands for $\sqrt{-1}$.

Solution of the equation $x^2 + 9 = 0$ leads to the complex number $x = 3i$ The real part of x is 0.

What complex number is obtained as solution of the equation, $2x^2 - 2x + 5 = 0$?

We will extend real arithmetic to complex numbers as follows:

If $z_1 = 1 - 2i$ and $z_2 = 2 + i$

then $\left.\begin{array}{l} z_1 + z_2 = 3 - i \\ z_1 - z_2 = -1 - 3i \end{array}\right\}$ add real and imaginary parts separately

$$z_1 \times z_2 = (1-2i)(2+i) = 1 \times 2 + 1 \times i - 2i \times 2 - 2i^2$$
$$= 2 - 3i - 2 \times -1 = 4 - 3i$$

(multiply as usual and put $i \times i = i^2 = -1$)

$$z_1/z_2 = \frac{1-2i}{2+i} = \frac{1-2i}{2+i}\,\frac{2-i}{2-i} = \frac{2 - i - 4i + 2i^2}{4 - i^2} = \frac{-5i}{5} = -i$$

$2-i$ is said to be the complex conjugate of $2+i$

In general if $z = a + ib$, its complex conjugate,

$$z^* = a - ib$$

$$z\,z^* = (a+ib)(a-ib) = a^2 + b^2$$

$\sqrt{zz^*}$ is called the amplitude of the complex number z, and it is a ~~always~~ real quantity, written as $|z|$.

If $z_1 = 3 - i$, $z_2 = -2 + 4i$, find, $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, z_1/z_2, z_2/z_1, $|z_1|$, and $|z_2|$.

Let us see how one handles things like e^{pi}. Since it is a complex quantity we will write $e^{pi} = x + iy$, and we immediately have $x^2 + y^2 = 1$ **(how?)** . e^{pi} (or any other similar exponential number, eg. 10^{pi}) will have unit amplitude for any value of p.

Since $x^2 + y^2 = 1$, we may propose, $e^{pi} = x + iy = \cos\theta + i\sin\theta$ and investigate the relationship of p to θ. We will do this using simple arithmatic:

We write $e^{\frac{\pi}{2}\cdot\frac{1}{512}i} = e^{0.00306796i} = 1 + 0.00306796\,i$

(recall, $e^p \sim 1 + p$, when p is small)

By repeatedly squaring and writing the result as x + iy, complete the following table.

e^{pi}	$x + iy$
$e^{0.00306796i}$	$1 + 0.00306796\,i$
$e^{0.00613592i}$	$0.99999060 + 0.00613592\,i$
$e^{0.01227184i}$	$0.99994350 + 0.01227173\,i$
$e^{0.02454369i}$	$0.99973640 + 0.02454207\,i$
$e^{0.04908738i}$	$0.99887050 + 0.04907120\,i$
$e^{0.09817475i}$	$0.99533430 + 0.09803154\,i$
$e^{0.19634950i}$	
$e^{0.39269900i}$	
$e^{0.78539800i}$	
$e^{1.57079600i}$	
$e^{3.14159200i}$	
$e^{6.28318400i}$	

Each line is obtained from the previous line by squaring. (retain at least six significant digits)

continue and complete the table.

$(x+iy)^2 = x^2 - y^2 + 2ixy$

This table demonstrates that (within the precision of your arithmatic) (i) $e^{(\pi/2)i} = i$ and (ii) x and y are periodic functions of p with period 2π. Which means, in our earlier proposal, we can equate p with θ. This is a very important result: $\underline{e^{i\theta} = \cos\theta + i\sin\theta}$

Thus, we have two equivalent representation for a complex number: $z = a + ib = re^{i\theta}$

Express r and θ in terms of a and b.

$r =$ ______

$\theta =$ ______

Matrices

These are arrays of numbers with arithmatic operations defined for them.

example: $[A] = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ a 3×3 matrix has 3 rows and 3 columns. It is a square matrix (no. of rows = no. of columns)

$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ is a square matrix of order 2.

$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ is a 4×1 matrix: 4 rows and 1 column. Often referred to as a column vector. $\begin{bmatrix} 4 & -1 & 3 \end{bmatrix}$ is a row vector having 3 elements (1×3 matrix)

Addition/subtraction defined for two matrices having identical dimensions. Eg: $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+1 \\ 1+2 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 3 \end{bmatrix}$

i.e., if $[C] = [A] + [B]$, then the elements of $[C]$ are found as $c_{ij} = a_{ij} + b_{ij}$

Multiplication is defined for $[A]$ and $[B]$ only if the number of columns of $[A]$ is equal to the number of rows of $[B]$.

Examples: $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1\times4 + 2\times5 \\ 3\times4 + 1\times5 \end{bmatrix} = \begin{bmatrix} 14 \\ 17 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 1\times0 + 2\times4 & 1\times1 + 2\times0 \\ 3\times0 + 4\times4 & 3\times1 + 4\times0 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 16 & 3 \end{bmatrix}$

i.e., if $[C] = [A][B]$ where $[A]$ is a $l \times n$ matrix and $[B]$ is a $n \times m$ matrix, then $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Do the following arithmetic.

$[A] = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 1 \\ 1 & 4 & -1 \end{bmatrix}$ $[B] = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 4 \\ 2 & 1 & 0 \end{bmatrix}$, find (i) $[A] - [B]$
(ii) $[A][B]$

$[P] = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}$ $[Q] = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ find (i) $[A][B]$
(ii) $[B][A]$

A number called <u>determinant</u> is defined for square matrices. Its calculation is illustrated below.

$[A] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\det[A] = |A| = 1\times 4 - 2\times 3 = -2$

$$[A] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \det[A] = |A| = 1\times\{5\times 9 - 6\times 8\} - 2\times\{4\times 9 - 6\times 7\} + 3\times\{4\times 8 - 5\times 7\}$$

$$= 1\times -3 - 2\times -6 + 3\times -3 = -3+12-9 = 0$$

It can also be calculated as $|A| = 1\times\{5\times 9 - 6\times 8\} - 4\times\{2\times 9 - 3\times 8\} + 7\times\{2\times 6 - 3\times 5\}$

$$= 1\times -3 - 4\times -6 + 7\times -3 = -3 + 24 - 21 = 0$$

A matrix like this ~~to~~ one, whose determinant is zero is called a singular matrix.

Evaluate the determinants: (i) $\begin{vmatrix} 8 & 0 & 5 \\ 2 & 1 & 4 \\ 0 & 1 & 5 \end{vmatrix}$, (ii) $\begin{vmatrix} 2 & 1 & 2 \\ 1 & 3 & 4 \\ 2 & 1 & 2 \end{vmatrix}$

Express the result for $\vec{a} \times \vec{b}$ (p.9) in the form of a determinant

<u>Special Square matrices</u>:

$[A]$ is a symmetric matrix if $a_{ij} = a_{ji}$ eg. $\begin{bmatrix} 3 & 4 \\ 4 & 8 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & -1 \\ 0 & -1 & 3 \end{bmatrix}$.

$[A]$ is a diagonal matrix if $a_{ij} = 0$ when $i \neq j$

eg: $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

$[A]$ is an identity matrix or unit matrix if it is diagonal and if all the diagonal elements are unity, and is written as $[I]$

Write down the unit matrix of order 4.

If $[A][B] = [I]$, then $[B]$ is ~~the~~ said to be the inverse of $[A]$ and vice versa ie, $[A]^{-1} = [B]$, $[B]^{-1} = [A]$.

Singular matrices will have no inverses.

$[A]$ is said to be the transpose of $[B]$ if $a_{ij} = b_{ji}$

eg: $[A] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, transpose of $[A] = [A]' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

[A] is said to be an orthogonal matrix if the following conditions are satisfied : $\sum_{k=1}^{n} a_{ik}a_{jk} = 1$ if $i=j$, $= 0$ if $i \neq j$; $\sum_{k=1}^{n} a_{ki}a_{kj} = 1$ if $i=j$, $=0$ if $i \neq j$

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $\begin{bmatrix} 0.6636 & 0.3514 & 0.6604 \\ -0.6104 & 0.7648 & 0.2063 \\ -0.4326 & -0.5400 & 0.7220 \end{bmatrix}$

Verify that the above matrices are orthogonal. Also verify that their transposes are their own inverses. This is generally true for all orthogonal matrices ($A^{-1} = A'$). ie., for an orthogonal matrix $[A][A'] = [I]$.

If [T] is an orthogonal matrix, the triple product $[T]'[A][T]$ is called an orthogonal transformation of [A].

In the following orthogonal transformation, find the value of θ, that will make [C] a diagonal matrix.

$$[C] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(Multiply and equate the off-diagonal elements to zero).

Expand the following determinant as a quadratic equation and solve for λ.

$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$ This is known as the characteristic equation of the matrix $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. For this case it is a quadratic equation. (For an n^{th}-order matrix the characteristic equation will be of degree n.) λ's are called eigenvalues.

Verify that the solutions you get for λ ~~is~~ are the same as the diagonal elements, you got for [C], above.

(Real symmetric matrices can be "diagonalised" by an orthogonal transformation The diagonal values are the eigen values.)

Finally, the elements of a matrix can be complex numbers. A square matrix is said to be Hermitian, if $a_{ij} = a_{ji}^*$

Example : $\begin{bmatrix} 1 & 1-i & 2+3i \\ 1+i & 2 & i \\ 2-3i & -i & 0 \end{bmatrix}$ Note that (i) the diagonal elements are real, (ii) real symmetric matrix is a special case of Hermitian matrix.